Dividing with variables and exponents

Topic TextTopic TermsGlossarySpanish Text Product and Quotient Rules Use the product rule to multiply exponential expressions Use the quotient rule to divide exponential expressions The Power Rule for Exponents Use the power rule to simplify expressions involving products, quotients, and exponents Negative and Zero Exponents Define and use the zero exponent rule Define and use the negative exponent rule Simplify Expressions Using the Exponent Rules Simplify expressions using a combination of the exponent rules Simplify compound exponential expressions with negative exponents Repeated Image We use exponential notation to write repeated multiplication. For example [latex]10\cdot10\cdot10[/latex] can be written more succinctly as [latex]10^{3}[/latex]. The 10 in [latex]10^{3}[/latex] is called the base. The 3 in [latex]10^{3}[/latex] is called the exponent. The expression [latex]10^{3}[/latex] is called the exponential expression. Knowing the names for the parts of an exponential expression or term will help you learn how to perform mathematical operations on them. [latex]\text{base}\rightarrow10^{3\leftarrow\text{exponent}}[/latex] [latex]10^{3}[/latex] is read as "10 to the third power" or "10 cubed." It means [latex]10\cdot10\cdot10[/latex], or 1,000. [latex]8^{2}[/latex] is read as "8 to the second power" or "8 squared." It means [latex]8\cdot8[/latex], or 64. [latex]5^{4}[/latex] is read as "5 to the fourth power." It means [latex]5\cdot5\cdot5\cdot5[/latex], or 625. [latex]b^{5}[/latex] is read as "b to the fifth power." It means [latex]{b}\cdot{b}\cdot{b}\cdot{b}\cdot{b}[/latex]. Its value will depend on the value of b. The exponent applies only to the number that it is next to. Therefore, in the expression [latex]xy^{4}[/latex], only the y is affected by the 4. [latex]xy^{4}[/latex] means [latex]{x}\cdot{y}\cdot{y}\cdot{y}\cdot{y}[/latex]. The x in this term is a coefficient of y. If the exponential expression is negative, such as [latex]−3^{4}[/latex], it means [latex]–\left(3\cdot3\cdot3\cdot3\right)[/latex] or [latex]−81[/latex]. If [latex]−3[/latex] is to be the base, it must be written as [latex]\left(−3\right)^{4}[/latex], which means [latex]−3\cdot−3\cdot−3\cdot−3[/latex], or 81. Likewise, [latex]\left(−x\right)^{4}=\left(−x\right)\cdot\left(−x\right)\cdot\left(−x\right)\cdot\left(−x\right)=x^{4}[/latex], while [latex]−x^{4}=–\left(x\cdot x\cdot x\cdot x\right)[/latex]. You can see that there is quite a difference, so you have to be very careful! The following examples show how to identify the base and the exponent, as well as how to identify the expanded and exponential format of writing repeated multiplication. Identify the exponent and the base in the following terms, then simplify: [latex]7^{2}[/latex] [latex]{\left(\frac{1}{2}\right)}^{3}[/latex] [latex]2x^{3}[/latex] [latex]\left(-5\right)^{2}[/latex] In the following video you are provided more examples of applying exponents to various bases. Evaluate expressions Evaluating expressions containing exponents is the same as evaluating the linear expressions from earlier in the course. You substitute the value of the variable into the expression and simplify. You can use the order of operations to evaluate the expressions containing exponents. First, evaluate anything in Parentheses or grouping symbols. Next, look for Exponents, followed by Multiplication and Division (reading from left to right), and lastly, Addition and Subtraction (again, reading from left to right). So, when you evaluate the expression [latex]5x^{3}[/latex] if [latex]x=4[/latex], first substitute the value 4 for the variable x. Then evaluate, using order of operations. In the example below, notice the how adding parentheses can change the outcome when you are simplifying terms with exponents. The addition of parentheses made quite a difference! Parentheses allow you to apply an exponent to variables or numbers that are multiplied, divided, added, or subtracted to each other. Caution! Whether to include a negative sign as part of a base or not often leads to confusion. To clarify whether a negative sign is applied before or after the exponent, here is an example. What is the difference in the way you would evaluate these two terms? [latex]-{3}^{2}[/latex] [latex]{\left(-3\right)}^{2}[/latex] To evaluate 1), you would apply the exponent to the three first, then apply the negative sign last, like this: [latex]\begin{array}{c}-\left({3}^{2}\right)\\=-\left(9\right) = -9\end{array}[/latex] To evaluate 2), you would apply the exponent to the 3 and the negative sign: [latex]\begin{array}{c}{\left(-3\right)}^{2}\\=\left(-3\right)\cdot\left(-3\right)\\={ 9}\end{array}[/latex] The key to remembering this is to follow the order of operations. The first expression does not include parentheses so you would apply the exponent to the integer 3 first, then apply the negative sign. The second expression includes parentheses, so hopefully you will remember that the negative sign also gets squared. In the next sections, you will learn how to simplify expressions that contain exponents. Come back to this page if you forget how to apply the order of operations to a term with exponents, or forget which is the base and which is the exponent! In the following video you are provided with examples of evaluating exponential expressions for a given number. Use the product rule to multiply exponential expressions Exponential notation was developed to write repeated multiplication more efficiently. There are times when it is easier or faster to leave the expressions in exponential notation when multiplying or dividing. Let's look at rules that will allow you to do this. For example, the notation [latex]5^{4}[/latex] can be expanded and written as [latex]5\cdot5\cdot5\cdot5[/latex], or 625. And don't forget, the exponent only applies to the number immediately to its left, unless there are parentheses. What happens if you multiply two numbers in exponential form with the same base? Consider the expression [latex]{2}^{3}{2}^{4}[/latex]. Expanding each exponent, this can be rewritten as [latex]\left(2\cdot2\cdot2\right)\left(2\cdot2\cdot2\cdot2\right)[/latex] or [latex]2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2[/latex]. In exponential form, you would write the product as [latex]2^{7}[/latex]. Notice that 7 is the sum of the original two exponents, 3 and 4. What about [latex]{x}^{2}{x}^{6}[/latex]? This can be written as [latex]\left(x\cdot{x}\right)\left(x\cdot{x}\cdot{x}\cdot{x}\cdot{x}\cdot{x}\right)=x\cdot{x}\cdot{x}\cdot{x}\cdot{x}\cdot{x}\cdot{x}\cdot{x}[/latex] or [latex]x^{8}[/latex]. And, once again, 8 is the sum of the original two exponents. This concept can be generalized in the following way: For any number x and any integers a and b, [latex]\left(x^{a}\right)\left(x^{b}\right) = x^{a+b}[/latex]. To multiply exponential terms with the same base, add the exponents. Caution! When you are reading mathematical rules, it is important to pay attention to the conditions on the rule. For example, when using the product rule, you may only apply it when the terms being multiplied have the same base and the exponents are integers. Conditions on mathematical rules are often given before the rule is stated, as in this example it says "For any number x, and any integers a and b." When multiplying more complicated terms, multiply the coefficients and then multiply the variables. Caution! Do not try to apply this rule to sums. Think about the expression [latex]\left(2+3\right)^{2}[/latex] Does [latex]\left(2+3\right)^{2}[/latex] equal [latex]2^{2}+3^{2}[/latex]? No, it does not because of the order of operations! [latex]\left(2+3\right)^{2}=5^{2}=25[/latex] and [latex]2^{2}+3^{2}=4+9=13[/latex] Therefore, you can only use this rule when the numbers inside the parentheses are being multiplied (or divided, as we will see next). Use the quotient rule to divide exponential expressions Let's look at dividing terms containing exponential expressions. What happens if you divide two numbers in exponential form with the same base? Consider the following expression. [latex] \displaystyle \frac{{{4}^{5}}}{{{4}^{2}}}[/latex] You can rewrite the expression as: [latex] \displaystyle \frac{4\cdot 4\cdot 4\cdot 4\cdot 4}{4\cdot 4}[/latex]. Then you can cancel the common factors of 4 in the numerator and denominator: [latex] \displaystyle [/latex] Finally, this expression can be rewritten as [latex]4^{3}[/latex] using exponential notation. Notice that the exponent, 3, is the difference between the two exponents in the original expression, 5 and 2. So, [latex] \displaystyle \frac{{{4}^{5}}}{{{4}^{2}}}=4^{5-2}=4^{3}[/latex]. Be careful that you subtract the exponent in the denominator from the exponent in the numerator. So, to divide two exponential terms with the same base, subtract the exponents. For any non-zero number x and any integers a and b: [latex] \displaystyle \frac{{{x}^{a}}}{{{x}^{b}}}={{x}^{a-b}}[/latex] When dividing terms that also contain coefficients, divide the coefficients and then divide variable powers with the same base by subtracting the exponents. In the following video we show another example of how to use the quotient rule to divide exponential expressions Raise powers to powers Another word for exponent is power. You have likely seen or heard an example such as [latex]3^5[/latex] can be described as 3 raised to the 5th power. In this section we will further expand our capabilities with exponents. We will learn what to do when a term with a power is raised to another power, and what to do when two numbers or variables are multiplied and both are raised to an exponent. We will also learn what to do when numbers or variables that are divided are raised to a power. We will begin by raising powers to powers. Let's simplify [latex]\left(5^{2}\right)^{4}[/latex]. In this case, the base is [latex]5^2[/latex] and the exponent is 4, so you multiply [latex]5^{2}[/latex] four times: [latex]\left(5^{2}\right)^{4}=5^{2}\cdot5^{2}\cdot5^{2}\cdot5^{2}=5^{8}[/latex] (using the Product Rule—add the exponents). [latex]\left(5^{2}\right)^{4}[/latex] is a power of a power. It is the fourth power of 5 to the second power. And we saw above that the answer is [latex]5^{8}[/latex]. Notice that the new exponent is the same as the product of the original exponents: [latex]2\cdot4=8[/latex]. So, [latex]\left(5^{2}\right)^{4}=5^{2\cdot4}=5^{8}[/latex] (which equals 390,625, if you do the multiplication). Likewise, [latex]\left(x^{4}\right)^{3}=x^{4\cdot3}=x^{12}[/latex] This leads to another rule for exponents—the Power Rule for Exponents. To simplify a power of a power, you multiply the exponents, keeping the base the same. For example, [latex]\left(2^{3}\right)^{5}=2^{15}[/latex]. For any positive number x and integers a and b: [latex]\left(x^{a}\right)^{b}=x^{a\cdot{b}}[/latex]. Take a moment to contrast how this is different from the product rule for exponents found on the previous page. Raise a product to a power Simplify this expression. [latex]\left(2a\right)^{4}=\left(2a\right)\left(2a\right)\left(2a\right)\left(2a\right)=\left(2\cdot2\cdot2\cdot2\right)\left(a\cdot{a}\cdot{a}\cdot{a}\cdot{a}\right)=\left(2^{4}\right)\left(a^{4}\right)=16a^{4}[/latex] Notice that the exponent is applied to each factor of 2a. So, we can eliminate the middle steps. [latex]\begin{array}{l}\left(2a\right)^{4} = \left(2^{4}\right)\left(a^{4}\right)\text{, applying the }4\text{ to each factor, }2\text{ and }a\\\\\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,=16a^{4}\end{array}[/latex] The product of two or more numbers raised to a power is equal to the product of each number raised to the same power. For any nonzero numbers a and b and any integer x, [latex]\left(ab\right)^{x}=a^{x}\cdot{b^{x}}[/latex]. How is this rule different from the power raised to a power rule? How is it different from the product rule for exponents on the previous page? If the variable has an exponent with it, use the Power Rule: multiply the exponents. Raise a quotient to a power Now let's look at what happens if you raise a quotient to a power. Remember that quotient means divide. Suppose you have [latex] \displaystyle \frac{3}{4}[/latex] and raise it to the 3rd power. [latex] \displaystyle {{\left(\frac{3}{4} \right)}^{3}}=\left(\frac{3}{4} \right)\left(\frac{3}{4} \right)\left(\frac{3}{4} \right)=\frac{3\cdot 3\cdot 3}{4\cdot 4\cdot 4}=\frac{{{3}^{3}}}{{{4}^{3}}}[/latex] You can see that raising the quotient to the power of 3 can also be written as the numerator (3) to the power of 3, and the denominator (4) to the power of 3. Similarly, if you are using variables, the quotient raised to a power is equal to the numerator raised to the power over the denominator raised to power. [latex] \displaystyle {{\left(\frac{a}{b} \right)}^{4}}=\left(\frac{a}{b} \right)\left(\frac{a}{b} \right)\left(\frac{a}{b} \right)\left(\frac{a}{b} \right)=\frac{a\cdot a\cdot a\cdot a}{b\cdot b\cdot b\cdot b}=\frac{{{a}^{4}}}{{{b}^{4}}}[/latex] When a quotient is raised to a power, you can apply the power to the numerator and denominator individually, as shown below. [latex] \displaystyle {{\left(\frac{a}{b} \right)}^{4}}=\frac{{{a}^{4}}}{{{b}^{4}}}[/latex] For any number a, any non-zero number b, and any integer x, [latex] \displaystyle {\left(\frac{a}{b}\right)}^{x}=\frac{a^{x}}{b^{x}}[/latex]. In the following video you will be shown examples of simplifying quotients that are raised to a power. Define and use the zero exponent rule When we defined the quotient rule, we only worked with expressions like the following: [latex]\frac{{{4}^{9}}}{{{4}^{4}}}[/latex], where the exponent in the numerator (up) was greater than the one in the denominator (down), so the final exponent after simplifying was always a positive number, and greater than zero. In this section, we will explore what happens when we apply the quotient rule for exponents and get a negative or zero exponent. What if the exponent is zero? To see how this is defined, let us begin with an example. We will use the idea that dividing any number by itself gives a result of 1. [latex]\frac{t^{8}}{t^{8}}=\frac{\cancel{t^{8}}}{\cancel{t^{8}}}=1[/latex] If we were to simplify the original expression using the quotient rule, we would have [latex]\frac{{t}^{8}}{{t}^{8}}={t}^{8 - 8}={t}^{0}[/latex] If we equate the two answers, the result is [latex]{t}^{0}=1[/latex]. This is true for any nonzero real number, or any variable representing a real number. [latex]{a}^{0}=1[/latex] The sole exception is the expression [latex]{0}^{0}[/latex]. This appears later in more advanced courses, but for now, we will consider the value to be undefined, or DNE (Does Not Exist). Any number or variable raised to a power of 1 is the number itself. [latex]n^{1}=n[/latex] Any non-zero number or variable raised to a power of 0 is equal to 1 [latex]n^{0}=1[/latex] The quantity [latex]0^{0}[/latex] is undefined. As done previously, to evaluate expressions containing exponents of 0 or 1, substitute the value of the variable into the expression and simplify. In the following video there is an example of evaluating an expression with an exponent of zero, as well as simplifying when you get a result of a zero exponent. Define and use the negative exponent rule We proposed another question at the beginning of this section. Given a quotient like [latex] \displaystyle \frac{{{2}^{m}}}{{{2}^{n}}}[/latex] what happens when n is larger than m? We will need to use the negative rule of exponents to simplify the expression so that it is easier to understand. Let's look at an example to clarify this idea. Given the expression: [latex]\frac{{h}^{3}}{{h}^{5}}[/latex] Expand the numerator and denominator, all the terms in the numerator will cancel to 1, leaving two hs multiplied in the denominator, and a numerator of 1. [latex]\begin{array}{l} \frac{{h}^{3}}{{h}^{5}}\,\,\,=\,\,\,\frac{h\cdot{h}\cdot{h}}{h\cdot{h}\cdot{h}\cdot{h}\cdot{h}} \\ \,\,\,\,\,\,\,\,\,\,\,\,\,=\,\,\,\frac{\cancel{h}\cdot \cancel{h}\cdot \cancel{h}}{\cancel{h}\cdot \cancel{h}\cdot \cancel{h}\cdot {h}\cdot {h}}\\\,\,\,\,\,\,\,\,\,\,\,\,\,=\,\,\,\frac{1}{h\cdot{h}}\\\,\,\,\,\,\,\,\,\,\,\,\,\,=\,\,\,\frac{1}{{h}^{2}} \end{array}[/latex] We could have also applied the quotient rule from the last section, to obtain the following result: [latex]\begin{array}{r}\frac{h^{3}}{h^{5}}\,\,\,=\,\,\,h^{3-5}\\\\=\,\,\,h^{-2}\,\,\end{array}[/latex] Putting the answers together, we have [latex]{h}^{-2}=\frac{1}{{h}^{2}}[/latex]. This is true when h, or any variable, is a real number and is not zero. For any nonzero real number [latex]a[/latex] and natural number [latex]n[/latex], the negative rule of exponents states that [latex]{a}^{-n}=\frac{1}{{a}^{n}}[/latex] Let's looks at some examples of how this rule applies under different circumstances.